

Positive Solutions for the p -Laplacian with Dependence on the Gradient

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Abstract

We prove a result of existence of positive solutions of the Dirichlet problem for $-\Delta_p u = w(x)f(u, \nabla u)$ in a bounded domain $\Omega \subset \mathbb{R}^N$, where Δ_p is the p -Laplacian and w is a weight function. As in previous results by the authors, and in contrast with the hypotheses usually made, no asymptotic behavior is assumed on f , but simple geometric assumptions on a neighborhood of the first eigenvalue of the p -Laplacian operator. We start by solving the problem in a radial domain by applying the Schauder Fixed Point Theorem and this result is used to construct an ordered pair of sub- and super-solution, also valid for nonlinearities which are super-linear both at the origin and at $+\infty$. We apply our method to the Dirichlet problem $-\Delta_p u = \lambda u(x)^{q-1}(1 + |\nabla u(x)|^p)$ in Ω and give examples of super-linear nonlinearities which are also handled by our method.

1 Introduction

It is usually said that the sub- and super-solution method does not handle problems which are superlinear at the origin. One of the main purposes of this paper is to prove that this is not true.

Furthermore, considering the eigenvalues of the natural operator defined by the equation, we believe that imposing asymptotic conditions on the nonlinearity masks one of the main problems in differential equations, which is to completely understand how the crossing of the eigenvalues by the nonlinearity determines the solutions of the equation. This paper presents a contribution in this direction.

For this, we consider the Dirichlet problem

$$\begin{cases} -\Delta_p u &= \omega(x)f(u, |\nabla u|) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ ($N > 1$) is a smooth, bounded domain, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian, $1 < p < \infty$, $\omega: \overline{\Omega} \rightarrow \mathbb{R}$ is a continuous, nonnegative function

*The authors were supported in part by FAPEMIG and CNPq-Brazil.

with isolated zeros (which we will call *weight function*) and the C^1 -nonlinearity $f: [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ satisfies simple hypotheses.

We solve (1) for a large class of functions f , including nonlinearities that are super-linear both at the origin and at $+\infty$. (The continuous function ω has isolated zeroes only to simplify the presentation. It is enough that $\omega(x_0) > 0$ for some $x_0 \in \Omega$.)

We apply our approach to prove the existence of positive solution for the problem

$$\begin{cases} -\Delta_p u &= \lambda u(x)^{q-1}(1 + |\nabla u(x)|^p) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

and, in the sequence, we give examples of super-linear nonlinearities (both at the origin and at $+\infty$) which are also handled by our method.

Adapting methods and techniques developed in [9], where the nonlinearity does not depend on ∇u , we start by obtaining a radial, positive solution u for the problem

$$\begin{cases} -\Delta_p u &= \omega_\rho(|x - x_0|)f(u, |\nabla u|) & \text{in } B_\rho, \\ u &= 0 & \text{on } \partial B_\rho, \end{cases} \quad (2)$$

where B_ρ is the ball with radius ρ centered at x_0 and ω_ρ a *radial* weight function. For this, no asymptotic behavior on f is assumed but, instead, simple local hypotheses on the nonlinearity f . (See hypotheses (H1) and (H2) in the sequence.) The application of the Schauder Fixed Point Theorem yields a radial solution u of (2).

To cope with the general case of a smooth, bounded domain Ω , we apply the method of sub- and super-solution as developed in [3] (see Theorem 2 in the next section), for the general problem

$$\begin{cases} -\Delta_p u &= f(x, u, |\nabla u|) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \quad (3)$$

where $f: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$, $(x, u, v) \mapsto f(x, u, v)$ is a Carathéodory function (i.e., measurable in the x -variable and continuous in the (u, v) -variable) satisfying

(H3) $f(x, u, v) \leq C(|u|)(1 + |v|^p)$ $(u, v) \in \mathbb{R} \times \mathbb{R}^N$, *a.e.* $x \in \Omega$ for some increasing function $C: [0, \infty] \rightarrow [0, \infty]$.

We observe that (H3) is also found in papers that do not apply the sub- and super-solution method (see [11, 25]), since they are also related to the regularity of a weak solution.

Besides the Bernstein-Nagumo type assumption (H3), our hypotheses on the nonlinearity f are not usual in the literature: we assume that f has a *local* behavior satisfying hypotheses of the type

(H1) $0 \leq f(u, |v|) \leq k_1 M^{p-1}$, if $0 \leq u \leq M$, $|v| \leq \gamma M$,

(H2) $f(u, |v|) \geq k_2 \delta^{p-1}$, if $0 < \delta \leq u \leq M$, $|v| \leq \gamma M$,

where the constants k_1 , k_2 and γ are defined later on in this paper and δ, M are arbitrary. These constants depend strongly on the weight function ω and in some special cases (for example, $\omega \equiv 1$) can be explicitly calculated, see Subsection 6.1. In [5] was proved that $k_1 < \lambda_1 < k_2$, where λ_1 stands for the first eigenvalue of the p -Laplacian.

Hypotheses (H1) and (H2) are geometrically interpreted in Figure 1.

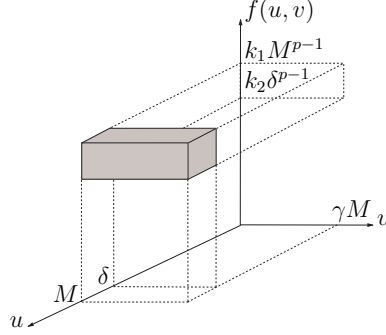


Figure 1: The graph of f stays below $k_1 M^{p-1}$ in $[0, M] \times [0, \gamma M]$ and passes through the gray box.

Hypotheses of this type will be considered in the scenarios of both the radial problem (2) and the general problem (1).

The radial problem (2) is solved as an application of the Schauder Fixed Point Theorem. Therefore, a hypothesis like (H3) is not needed while studying this problem.

By considering a ball $B_\rho \subset \Omega$, radial symmetrization of the weight function ω permits us to consider a problem in the radial form in the sub-domain B_ρ , which has a solution u_ρ as a consequence of our study of problem (2). The chosen ball B_ρ determines the value of the constants k_2 and γ needed to solve (1) and the radial solution $u_\rho: \overline{B_\rho} \rightarrow \mathbb{R}$ produces a sub-solution \underline{u} of problem (1), when we consider the extension \underline{u} of u_ρ defined by $\underline{u}(x) = 0$, if $x \in \Omega \setminus B_\rho$. So, the solution of (2) gives rise to a sub-solution of problem (1).

In order to obtain a super-solution \overline{u} for problem (1), we impose that

$$\frac{\|\nabla \overline{u}\|_\infty}{\|\overline{u}\|_\infty} \leq \gamma, \quad (4)$$

an estimate that is suggested by hypothesis (H1). So, we look for a super-solution of (1) satisfying (4) and defined in a (smooth, bounded) domain $\Omega_2 \supset \Omega$, which determines the value of the constant k_1 needed to solve (1).

In the abstract setting of the domain Ω_2 , the super-solution \overline{u} turns out to be a multiple of the solution ϕ_{Ω_2} of the problem

$$\begin{cases} -\Delta_p \phi_{\Omega_2} &= \|\omega\|_\infty & \text{in } \Omega_2, \\ \phi_{\Omega_2} &= 0 & \text{on } \partial\Omega_2, \end{cases} \quad (5)$$

if ϕ_{Ω_2} satisfies (4). In this setting, the existence of a positive solution for (1) is stated in Section 5.

We give two applications of this result for abstract nonlinearities in Section 6. In the first application, given in Subsection 6.2, we choose a ball $\Omega_2 = B_R$ such that $\Omega \subset B_R$ and prove that, if R is large enough, it is possible to obtain a super-solution for (1) satisfying (4).

The second application is more demanding and considers the case where Ω_2 is the domain Ω itself. In order to control the quotient (4), we assume Ω to be convex and apply a maximum result proved in Payne and Philippin [24]. But, in some cases, if we choose Ω_2 as the convex hull of Ω , the same method produces a better solution than considering $\Omega \subset B_R$ for R large enough.

In Section 7 we consider concrete nonlinearities f . For $\lambda \in (0, \lambda^*]$ (where λ^* is a positive constant), we apply the technique of Subsection 6.2 and prove the existence of a positive solution for the problem

$$\begin{cases} -\Delta_p u &= f_\lambda(u, |\nabla u|) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

where $f_\lambda(u, |\nabla u|) := \lambda u(x)^{q-1}(1 + |\nabla u(x)|^p)$, $1 < p < q$.

In the sequence, simple modifications in $f_\lambda(u, |\nabla u|)$ will produce new nonlinearities which are also handled by our method, including nonlinearities with superlinear behavior both at the origin and at $+\infty$.

2 Comments on our method

In general, variational techniques are not suitable to handle (1); therefore, a combination of topological methods (as fixed-point or degree results) and blow-up arguments are usually applied to solve it ([11, 16, 25]). In the Laplacian case $p = 2$, a combination of the mountain pass geometry with the contraction lemma was first used in [10]: an iteration process is constructed by freezing the gradient in each iteration and (variationally) solving the resulting problem. Then, Lipschitz hypotheses in the variables u and v are made on $f(x, u, v)$ in order to guarantee the convergence in $W_0^{1,2}(\Omega)$ of the obtained sequence of solutions. The same approach for the p -Laplacian with $p > 2$ is not directly adaptable, since the natural extension of the Lipschitz conditions used to obtain the convergence of the iterated solutions yield a Hölder function f with exponent greater than 1 in variables u and v .

When the nonlinearity f does not depend on the gradient, the same technique was generalized in [5] to a smooth, bounded domain $\Omega \subset \mathbb{R}^N$. However, if Ω is not a ball, the dependence of f on $|\nabla u|$ demands controlling $\|\nabla u\|_\infty$ in Ω and complicates the application of Schauder's Fixed Point Theorem.

In [11], the authors discuss the existence of positive solutions for quasilinear elliptic equations in annular domains in \mathbb{R}^N and, in particular, the radial Dirichlet problem in annulus. (Therefore, the problem is transformed into an ordinary differential equation.) In that paper, f satisfies a super-linear condition at 0 and a local super-linear condition at $+\infty$. The growth of the nonlinearity f in

relation to the gradient is controlled by a hypothesis similar to (H3) and a local homogeneity type condition in the second variable, hypothesis related to the behavior of f near a point (r, s, t) such that $f(r, s, t) = 0$, where $r = |x|$. The existence of solutions is guaranteed by applying the Krasnosel'skii Fixed Point Theorem for mappings defined in cones.

The majority of papers dealing with the sub- and super-solution method with nonlinearities depending on the gradient are focused in the improvement of the method itself, that is, the papers aim to weaken the hypotheses of the method ([19, 20]). One exception is the paper of Grenon [14], where problem (3) is solved by analyzing two symmetrized problems. From the existence of two nontrivial super-solutions V_1 and V_2 for those problems follows the existence of a super-solution U_1 and a sub-solution U_2 for (3), with $U_2 \leq U_1$.

More recently, the sub- and super-solution method has been applied to some instances of problem (1): the Laplacian case $p = 2$ and $\Omega = \mathbb{R}^N$. In [12, 13], the dependence on the gradient occurs by means of a convection term $|\nabla u|^\alpha$ in the nonlinearity f and the authors look for ground state solutions. These are obtained as limits of a monotone sequence of auxiliary problems defined in nested subdomains of \mathbb{R}^n , which are bounded and smooth.

3 Preliminaries

In this section we recall some basic results in the theory of the p -Laplacian equation with Dirichlet boundary condition and present technical results that will be used in the rest of the paper. Let D be a bounded, smooth domain in \mathbb{R}^N , $N > 1$.

Definition 1 *Let $f: D \times \mathbb{R} \times \mathbb{R}^N$ be a Carathéodory function. A function $u \in W^{1,p}(D) \cap L^\infty(D)$ is called a solution (sub-solution, super-solution) of*

$$\begin{cases} -\Delta_p u &= f(x, u, \nabla u) & \text{in } D, \\ u &= 0 & \text{on } \partial D, \end{cases} \quad (6)$$

if

$$\int_D |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = \int_D f(x, u, \nabla u) \phi \, dx \quad (\leq 0, \geq 0),$$

for all $\phi \in C_0^\infty(D)$, ($\phi \geq 0$ in D in the case of a sub- or super-solution) and

$$u = 0 \quad (\leq 0, \geq 0) \quad \text{on } \partial D.$$

A pair $(\underline{u}, \overline{u})$ of sub- and super-solution is ordered if $\underline{u} \leq \overline{u}$ a.e.

The hypothesis (H3) implies that

$$\begin{aligned} \int_D |f(x, u, \nabla u) \phi| \, dx &\leq C(\|u\|_\infty) \int_D (1 + |\nabla u|^p) |\phi| \, dx \\ &= C(\|u\|_\infty) \left(\int_D |\phi| \, dx + \int_D |\nabla u|^p |\phi| \, dx \right) < \infty, \end{aligned}$$

since $\phi \in C_0^\infty$ and $u \in W^{1,p}(D) \cap L^\infty(D)$.

We now state, in a version adapted to our paper, the result that give basis to the method of sub- and super-solution for equations like (6). The existence part is a consequence of Theorem 2.1 of Boccardo, Murat and Puel [3]. The regularity part follows from the estimates of Lieberman [21], while the minimal and maximal solutions are consequence of Zorn's Lemma, as proved in Cuesta Leon [7]:

Theorem 2 *Let $f: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a Carathéodory function satisfying (H3). Suppose that (\underline{u}, \bar{u}) is an ordered pair of sub- and super-solution for the problem (6).*

Then, there exists a minimal solution u and a maximal solution v of (6), both in $C^{1,\tau}(\bar{\Omega})$ ($0 < \tau < 1$), such that $\underline{u} \leq u \leq v \leq \bar{u}$.

(By *minimal* and *maximal* solution of (6) we mean that, if w is a solution of this problem and $\underline{u} \leq w \leq \bar{u}$, then $u \leq w \leq v$.)

Remark 3 It is well-known that, for each $h \in L^\infty(D)$, the problem

$$\begin{cases} -\Delta_p u &= h & \text{in } D, \\ u &= 0 & \text{on } \partial D \end{cases} \quad (7)$$

has a unique weak solution $u \in W_0^{1,p}(D)$, which belongs to $C^{1,\tau}(\bar{D})$ for some $0 < \tau < 1$.

If we assume that $f: \bar{D} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is continuous, then is compact and continuous the operator $T: C^1(\bar{D}) \rightarrow C^1(\bar{D})$ defined by $Tu = v$, where v is the solution of

$$\begin{cases} -\Delta_p v &= f(x, u, \nabla u) & \text{in } D, \\ v &= 0 & \text{on } \partial D. \end{cases} \quad (8)$$

(The space $C^1(\bar{D})$ is the Banach space of continuously differentiable functions endowed with the norm $\|u\|_1 = \|u\|_\infty + \|\nabla u\|_\infty$.)

Theorem 4 *T is continuous and compact.*

Proof. It is clear that the mapping $T_1: C^1(\bar{D}) \rightarrow L^\infty(\bar{D})$ given by $T_1(u) = f(x, u, \nabla u)$ is continuous. As pointed out in [2], Lemma 1.1, the L^∞ -estimates of Anane [1] and the $C^{1,\alpha}$ -estimates of Liebermann [21] and Tolksdorf [27] imply that $T_2: L^\infty(\bar{D}) \rightarrow C^1(\bar{D})$, defined by $T_2(h) = v$, where

$$\begin{cases} -\Delta_p v &= h & \text{in } D, \\ v &= 0 & \text{on } \partial D. \end{cases}$$

is compact and continuous.

Since $T = T_2 \circ T_1$, where are done. □

Problem (7) satisfies a comparison principle and a strong maximum principle. (See [8], Thm 1.2 and Thm 2.2, respectively.) It follows easily a comparison principle between solutions defined in different domains:

Lemma 5 Suppose that Ω_1, Ω_2 are smooth domains, $\Omega_1 \subset \Omega_2$. For $i \in \{1, 2\}$, let $L^\infty \ni h_i: \Omega_i \rightarrow \mathbb{R}$ and $u_i \in C^{1,\alpha}(\overline{\Omega_i})$ be the weak solution of the problem

$$\begin{cases} -\Delta_p u_i &= h_i & \text{in } \Omega_i, \\ u_i &= 0 & \text{on } \partial\Omega_i. \end{cases} \quad (9)$$

Let us suppose that

$$(i) \quad 0 \leq h_1 \leq h_2 \text{ in } \Omega_1,$$

$$(ii) \quad u_1 \leq u_2 \text{ in } \partial\Omega_1.$$

Then $u_1 \leq u_2$ in Ω_1 .

In the setting of equation (7), we define

$$k_1(D) := \|\phi_D\|_\infty^{-(p-1)} \quad (10)$$

where $\phi_D \in C^{1,\alpha}(\overline{D}) \cap W_0^{1,p}(D)$ is the solution of

$$\begin{cases} -\Delta_p \phi_D &= \omega_D & \text{in } D, \\ \phi_D &= 0 & \text{on } \partial D. \end{cases} \quad (11)$$

By the maximum principle, $\phi_D > 0$ in D and $k_1(D)$ is well defined.

Remark 6 In the context of Lemma 5, that is, $\Omega_1 \subset \Omega_2$ and $\omega_{\Omega_1} \leq \omega_{\Omega_2}$, it follows immediately that

$$k_1(\Omega_2) = \|\phi_{\Omega_2}\|_\infty^{-(p-1)} \leq \|\phi_{\Omega_1}\|_\infty^{-(p-1)} = k_1(\Omega_1).$$

In the special case $D = B_\rho$, a ball of radius ρ centered in $x_0 \in \Omega$, let us consider the Dirichlet problem

$$\begin{cases} -\Delta_p \phi_\rho &= \omega_\rho(|x - x_0|) & \text{in } B_\rho, \\ \phi_\rho &= 0 & \text{on } \partial B_\rho, \end{cases} \quad (12)$$

where $\omega_\rho: \overline{B_\rho} \rightarrow \mathbb{R}$ is a *radial* weight function.

It is straightforward to verify that the solution of (12) is given by

$$\phi_\rho(|x - x_0|) = \int_{|x-x_0|}^\rho \left(\int_0^\theta K(s, \theta) ds \right)^{\frac{1}{p-1}} d\theta, \quad |x - x_0| \leq \rho, \quad (13)$$

where

$$K(s, \theta) = \left(\frac{s}{\theta} \right)^{N-1} \omega_\rho(s). \quad (14)$$

The solution ϕ_ρ satisfies $\phi_\rho \in C^2(\overline{B_\rho})$ if $1 < p \leq 2$ and $\phi_\rho \in C^{1,\tau}(\overline{B_\rho})$ if $p > 2$, where $\tau = 1/(p-1)$. (See [4], Lemma 2 for details.)

We also define another constant that will play an essential role in our technique:

$$\begin{aligned} k_2(B_\rho) &= \left[\int_t^\rho \left(\int_0^t K(s, \theta) ds \right)^{\frac{1}{p-1}} d\theta \right]^{1-p} \\ &= \left[\max_{0 \leq r \leq \rho} \int_r^\rho \left(\int_0^r K(s, \theta) ds \right)^{\frac{1}{p-1}} d\theta \right]^{1-p}. \end{aligned} \quad (15)$$

Since ω_ρ has isolated zeroes and the function

$$\alpha \rightarrow \int_\alpha^\rho \left(\int_0^\alpha K(s, \theta) ds \right)^{\frac{1}{p-1}} d\theta$$

is nonnegative and vanishes both at $\alpha = 0$ and at $\alpha = \rho$, we have $t > 0$.

We now establish the relation between $k_1(D)$ and $k_2(B_\rho)$, also valid in the case $D = B_\rho$:

Lemma 7 *Let D be a smooth domain in \mathbb{R}^N ($N > 1$), $B_\rho \subseteq D$ a ball of center x_0 and radius $\rho > 0$ and $k_1(D)$, $k_2(B_\rho)$ the constants defined by (10) and (15), respectively, where ω_ρ is a radial weight function such that $\omega_D \geq \omega_\rho$ in B_ρ .*

Then, $k_1(D) < k_2(B_\rho)$.

Proof. We have

$$\begin{cases} -\Delta_p \phi_D = \omega_D & \geq \omega_\rho = -\Delta_p \phi_\rho & \text{in } B_\rho, \\ \phi_D & \geq 0 = \phi_\rho & \text{on } \partial B_\rho \end{cases}$$

and the comparison principle yields $\phi_D \geq \phi_\rho$ in B_ρ .

Therefore

$$\|\phi_D\|_\infty \geq \|\phi_\rho\|_\infty = \int_0^\rho \left(\int_0^\theta K(s, \theta) ds \right)^{\frac{1}{p-1}} d\theta$$

and

$$k_1(D)^{-1} = \|\phi_D\|_\infty^{p-1} \geq \|\phi_\rho\|_\infty^{p-1} = \left[\int_0^\rho \left(\int_0^\theta K(s, \theta) ds \right)^{\frac{1}{p-1}} d\theta \right]^{p-1}.$$

Since the zeroes of ω_D are isolated and $t \neq 0$, we have

$$k_1(D)^{-1} > \left[\int_t^\rho \left(\int_0^t K(s, \theta) ds \right)^{\frac{1}{p-1}} d\theta \right]^{p-1} = k_2(B_\rho)^{-1},$$

proving our result. \square

4 Radial Solutions

In this section we study the radial version of (1), that is

$$\begin{cases} -\Delta_p u &= \omega_\rho(|x - x_0|)f(u, |\nabla u|) & \text{in } B_\rho, \\ u &= 0 & \text{on } \partial B_\rho, \end{cases} \quad (2)$$

where B_ρ is a ball of radius ρ centered in x_0 and $\omega_\rho: \overline{B_\rho} \rightarrow \mathbb{R}$ is a radial weight function.

A solution of (2) will be obtained by applying the Schauder Fixed Point Theorem in the space $C^1(B_\rho)$. So, the hypothesis (H3) is not necessary; we only need f to be continuous.

The radial boundary value problem equivalent to (2) is

$$\begin{cases} \frac{d}{dr}(-r^{N-1}\varphi_p(u'(r))) &= r^{N-1}\omega_\rho(r)f(u, |u'(r)|), & 0 < r < \rho \\ u'(0) &= 0, \\ u(\rho) &= 0, \end{cases} \quad (16)$$

where $\varphi_p(\xi) = |\xi|^{p-2}\xi$ for $1 < p < \infty$.

If $q = p/p - 1$ and $u > 0$, the function φ_q , inverse of φ_p , is given by

$$\varphi_q(u) = |u|^{q-2}u = u^{q-1} = u^{\frac{p}{p-1}-1} = u^{\frac{1}{p-1}}.$$

We remark that the function $\phi_p(r)$, $r = |x - x_0|$, defined by (13), can be written as

$$\phi_p(r) = \int_r^\rho \varphi_p \left(\int_0^\theta K(s, \theta) ds \right) d\theta.$$

So,

$$\phi'_\rho(r) = -\varphi_q \left(\int_0^r K(s, r) ds \right)$$

and $|\nabla \phi_\rho(|x - x_0|)| = |\phi'_\rho(r) \frac{x - x_0}{r}| = |\phi'_\rho(r)|$. Therefore, we have $\|\nabla \phi_\rho\|_\infty = \max_{0 \leq r \leq \rho} |\phi'_\rho(r)|$.

To prove the existence of solutions for problem (2), we suppose the existence of δ and M , with $0 < \delta < M$ such that the nonlinearity f satisfies

$$(H1_r) \quad 0 \leq f(u, |v|) \leq k_1(B_\rho)M^{p-1}, \text{ if } 0 \leq u \leq M, |v| \leq \gamma_\rho M;$$

$$(H2_r) \quad f(u, |v|) \geq k_2(B_\rho)\delta^{p-1}, \text{ if } \delta \leq u \leq M, |v| \leq \gamma_\rho M,$$

with $k_1(B_\rho)$ and $k_2(B_\rho)$ defined by (10) and (15), respectively, and γ_ρ defined by

$$\gamma_\rho = \max_{0 \leq r \leq \rho} \varphi_q \left(k_1(B_\rho) \int_0^r K(s, r) ds \right) = \frac{\|\nabla \phi_\rho\|_\infty}{\|\phi_\rho\|_\infty}. \quad (17)$$

We remark that $k_1(B_\rho)$, $k_2(B_\rho)$ and γ_ρ depend only on ρ and ω_ρ . The hypothesis (H2_r) aims to discard $u \equiv 0$ as a solution of (2), in the case $f(0, |v|) = 0$.

We also define the continuous functions Ψ_δ , Φ_M and Γ_M by

$$\Psi_\delta(r) = \begin{cases} \delta, & \text{if } 0 \leq r \leq t, \\ \delta \int_r^\rho \varphi_q \left(k_2(B_\rho) \int_0^t K(s, \theta) ds \right) d\theta, & \text{if } t < r \leq \rho, \end{cases} \quad (18)$$

$$\begin{aligned} \Phi_M(r) &= M \int_r^\rho \varphi_q \left(k_1(B_\rho) \int_0^\theta K(s, \theta) ds \right) d\theta \\ &= M \frac{\phi_\rho(r)}{\|\phi_\rho\|_\infty}, \text{ if } 0 < r \leq \rho, \end{aligned} \quad (19)$$

and

$$\begin{aligned} \Gamma_M(r) &= M \varphi_q \left(k_1(B_\rho) \int_0^r K(s, r) ds \right) \\ &= M \frac{|\phi'_\rho(r)|}{\|\phi_\rho\|_\infty}, \text{ if } 0 < r \leq \rho. \end{aligned} \quad (20)$$

Lemma 8 *We have*

- (i) $0 \leq \Phi_M(r) \leq M$;
- (ii) $0 \leq \Gamma_M(r) \leq \gamma_\rho M$;
- (iii) $0 \leq \Psi_\delta(r) \leq \Phi_M(r)$.

Proof. The items (i) and (ii) are obvious.

It follows from (H1_r) and (H2_r) that $k_2(B_\rho)\delta^{p-1} \leq k_1(B_\rho)M^{p-1}$. Therefore, if $0 \leq r \leq t$, then

$$\begin{aligned} \Phi_M(r) &= M \int_r^\rho \varphi_q \left(k_1(B_\rho) \int_0^\theta K(s, \theta) ds \right) d\theta \\ &\geq \int_t^\rho \varphi_q \left(k_1(B_\rho)M^{p-1} \int_0^t K(s, \theta) ds \right) d\theta \\ &\geq \int_t^\rho \varphi_q \left(k_2(B_\rho)\delta^{p-1} \int_0^t K(s, \theta) ds \right) d\theta \\ &= \delta = \Psi_\delta(r). \end{aligned}$$

If $t < r \leq \rho$, then

$$\begin{aligned} \Phi_M(r) &= M \int_r^\rho \varphi_q \left(k_1(B_\rho) \int_0^\theta K(s, \theta) ds \right) d\theta \\ &\geq \int_r^\rho \varphi_q \left(k_2(B_\rho)\delta^{p-1} \int_0^\theta K(s, \theta) ds \right) d\theta \\ &\geq \delta \int_r^\rho \varphi_q \left(k_2(B_\rho) \int_0^t K(s, \theta) ds \right) d\theta = \Psi_\delta(r), \end{aligned}$$

completing the proof of (iii). \square

We now establish the main result of this section:

Theorem 9 *Suppose that the continuous nonlinearity f satisfies (H1_r) and (H2_r). Then the problem*

$$\begin{cases} -\Delta_p u &= \omega_\rho(|x - x_0|)f(u, |\nabla u|) & \text{in } B_\rho, \\ u &= 0 & \text{on } \partial B_\rho, \end{cases} \quad (2)$$

has at least one positive solution $u_\rho(|x - x_0|)$ satisfying

$$\Psi_\delta \leq u_\rho \leq \Phi_M \text{ and } |\nabla u_\rho| \leq \Gamma_M$$

(and so $\delta \leq \|u_\rho\|_\infty \leq M$ and $\|\nabla u_\rho\|_\infty \leq \gamma_\rho M$).

Proof. To obtain a positive solution of (16), we consider the Banach space $X = C^1([0, \rho])$, with the norm $\|u\| = \sup_{s \in [0, \rho]} |u(s)| + \sup_{s \in [0, \rho]} |u'(s)|$ and the integral operator $T: X \rightarrow X$ defined by

$$(Tu)(r) = \int_r^\rho \varphi_q \left(\int_0^\theta K(s, \theta) f(u(s), |u'(s)|) ds \right) d\theta, \quad 0 \leq r \leq \rho.$$

It follows immediately that

$$(Tu)'(r) = -\varphi_q \left(\int_0^r K(s, r) f(u(s), |u'(s)|) ds \right), \quad 0 \leq r \leq \rho.$$

Theorem 4 yields the continuity and compactness of T . (A direct proof that $T: X \rightarrow X$ is a continuous, compact operator can be found in the Appendix.)

Now we observe that, if u is a fixed point of the operator T , then u is a solution of (16). To prove the existence of a fixed point u of T , we apply the Schauder Fixed Point Theorem in the closed, convex and bounded subset

$$Y = \{u \in X : \Psi_\delta \leq u \leq \Phi_M \text{ and } |u'| \leq \Gamma_M\}.$$

We need only to show that $T(Y) \subset Y$.

It follows from Lemma 8 and (H1_r) that, for all $0 \leq r \leq \rho$, we have

$$\begin{aligned} (Tu)(r) &= \int_r^\rho \varphi_q \left(\int_0^\theta K(s, \theta) f(u(s), |u'(s)|) ds \right) d\theta \\ &\leq \int_r^\rho \varphi_q \left(k_1(B_\rho) M^{p-1} \int_0^\theta K(s, \theta) ds \right) d\theta \\ &= M \int_r^\rho \varphi_q \left(k_1(B_\rho) \int_0^\theta K(s, \theta) ds \right) d\theta \\ &= \Phi_M(r) \end{aligned}$$

and

$$\begin{aligned}
|(Tu)'(r)| &= \varphi_q \left(\int_0^r K(s, r) f(u(s), |u'(s)|) ds \right) \\
&\leq \varphi_q \left(k_1(B_\rho) M^{p-1} \int_0^r K(s, r) ds \right) \\
&= M \varphi_q \left(k_1(B_\rho) \int_0^r K(s, r) ds \right) \\
&= \Gamma_M(r) \leq \gamma_\rho M.
\end{aligned}$$

Suppose that $0 \leq r \leq t$. The definition of Y and (H2_r) imply that

$$f(u(s), |u'(s)|) \geq k_2(B_\rho) \delta^{p-1}.$$

Therefore,

$$\begin{aligned}
(Tu)(r) &= \int_r^\rho \varphi_q \left(\int_0^\theta K(s, \theta) f(u(s), |u'(s)|) ds \right) d\theta \\
&\geq \int_t^\rho \varphi_q \left(\int_0^t K(s, \theta) f(u(s), |u'(s)|) ds \right) d\theta \\
&\geq \int_t^\rho \varphi_q \left(k_2(B_\rho) \delta^{p-1} \int_0^t K(s, \theta) ds \right) d\theta \\
&= \delta = \Psi_\delta(r).
\end{aligned}$$

If $t \leq r \leq \rho$, then

$$\begin{aligned}
(Tu)(r) &= \int_r^\rho \varphi_q \left(\int_0^\theta K(s, \theta) f(u(s), |u'(s)|) ds \right) d\theta \\
&\geq \int_r^\rho \varphi_q \left(\int_0^t K(s, \theta) f(u(s), |u'(s)|) ds \right) d\theta \\
&\geq \int_r^\rho \varphi_q \left(k_2(B_\rho) \delta^{p-1} \int_0^t K(s, \theta) ds \right) d\theta \\
&= \Psi_\delta(r).
\end{aligned}$$

So, we have $T(Y) \subset Y$. By the Schauder Fixed Point Theorem, we conclude the existence of at least one positive solution u_ρ for (16) in Y , thus implying that $u_\rho(|x - x_0|)$ is a positive solution of (2) that satisfies the bounds stated in the theorem. \square

5 Existence of Solutions in General Domains

In this section we state and prove our main abstract result: the existence of a positive solution for

$$\begin{cases} -\Delta_p u &= \omega(x) f(u, |\nabla u|) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{cases} \quad (1)$$

We start by defining the parameters we need to formulate our hypotheses.
Let Ω_2 be a bounded, smooth domain such that $\Omega_2 \supset \Omega$ and define

$$k_1(\Omega_2) := \|\phi_{\Omega_2}\|_{\infty}^{1-p},$$

where ϕ_{Ω_2} is the solution of

$$\begin{cases} -\Delta_p \phi_{\Omega_2} &= \|\omega\|_{\infty} & \text{in } \Omega_2, \\ \phi_{\Omega_2} &= 0 & \text{on } \partial\Omega_2. \end{cases} \quad (21)$$

Now, for any ball $B_\rho \subset \Omega$ with center in $x_0 \in \Omega$ and radius $\rho > 0$, let us denote by ω_ρ the radial function defined by

$$\omega_\rho(s) = \begin{cases} \min_{|x-x_0|=s} \omega(x), & \text{if } 0 < s \leq \rho, \\ \omega(x_0), & \text{if } s = 0. \end{cases} \quad (22)$$

Thus, by using this function we consider $k_2(B_\rho)$ and γ_ρ , defined in accordance to the former definitions (15) and (17), respectively.

At last, we fix $\rho > 0$ such that (see Remark 11, below)

$$\frac{\|\nabla \phi_{\Omega_2}\|_{\infty}}{\|\phi_{\Omega_2}\|_{\infty}} \leq \gamma_\rho \quad (23)$$

and then we set the parameters

$$k_1 := k_1(\Omega_2), \quad k_2 := k_2(B_\rho) \quad \text{and} \quad \gamma = \gamma_\rho.$$

Theorem 10 *Suppose that, for arbitrary δ, M such that $0 < \delta < M$, the non-linearity f satisfies:*

$$(H1) \quad 0 \leq f(u, |v|) \leq k_1 M^{p-1}, \text{ if } 0 \leq u \leq M, |v| \leq \gamma M;$$

$$(H2) \quad f(u, |v|) \geq k_2 \delta^{p-1}, \text{ if } \delta \leq u \leq M, |v| \leq \gamma M;$$

$$(H3) \quad f(u, |v|) \leq C(|u|)(1 + |v|^p) \text{ for all } (x, u, v), \text{ where } C: [0, \infty) \rightarrow [0, \infty) \text{ is increasing.}$$

Then, problem (1) has a positive solution u such that

$$\delta \leq \|u\|_{\infty} \leq M \text{ in } \Omega.$$

Remark 11 We would like to observe that the inequality (23) always occurs, if ρ is taken sufficiently small such that

$$\frac{\|\nabla \phi_{\Omega_2}\|_{\infty}}{\|\phi_{\Omega_2}\|_{\infty}} \leq \frac{1}{\rho}. \quad (24)$$

In fact, we have the gross estimate

$$\frac{1}{\rho} \leq \gamma_\rho, \quad \text{for any } B_\rho \subset \Omega$$

since $\gamma_\rho = \frac{\|\nabla\phi_\rho\|_\infty}{\|\phi_\rho\|_\infty}$ and

$$\|\phi_\rho\|_\infty = \phi_\rho(0) = - \int_0^\rho \phi'_\rho(s) ds = \int_0^\rho |\phi'_\rho(s)| ds \leq \rho \|\nabla\phi_\rho\|_\infty.$$

We supposed that the weight function ω has isolated zeroes. As mentioned, this assumption is not necessary: it will only be used in the discussion about the best possible choice for the constants k_1 and k_2 , which is done in Subsection 6.2.

In Section 6 we give examples of Ω_2 and ρ satisfying (23). There, we consider the cases $\Omega_2 = B_R \supset \Omega$ and, supposing Ω convex, $\Omega_2 = \Omega$ and present better estimates than (24) to choose ρ .

The obtention of a sub-solution for problem (1) is based on the following general result:

Lemma 12 *Let Ω and Ω_1 be smooth domains in \mathbb{R}^N ($N > 1$), with $\Omega_1 \subset \Omega$. Let $u_1 \in C^{1,\tau}(\overline{\Omega_1})$ be a positive solution of*

$$\begin{cases} -\Delta_p u_1 &= f_1(x, u_1, \nabla u_1) & \text{in } \Omega_1, \\ u_1 &= 0 & \text{on } \partial\Omega_1, \end{cases}$$

where the nonnegative nonlinearity f_1 is continuous.

Suppose also that the set

$$Z_1 = \{x \in \Omega_1 : \nabla u_1 = 0\}$$

is finite.

Then the extension

$$\underline{u}(x) = \begin{cases} u_1(x), & \text{if } x \in \overline{\Omega_1}, \\ 0, & \text{if } x \in \overline{\Omega} \setminus \Omega_1 \end{cases}$$

is a sub-solution of

$$\begin{cases} -\Delta_p u &= f(x, u, \nabla u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega \end{cases}$$

for all continuous nonlinearities $f \geq 0$ such that $f_1(x, u, \nabla u) \leq f(x, u, \nabla u)$ in Ω_1 .

Proof. This proposition is a consequence of the Divergence Theorem combined with the Hopf's Lemma (which states that $\frac{\partial u}{\partial \eta} < 0$ on $\partial\Omega_1$, if η denotes the unit outward normal to $\partial\Omega_1$, see [26], Lemma A.3). Really, if $\phi \in C_0^\infty(\Omega)$ and $\phi \geq 0$, by assuming (without loss of generality) that $Z_1 = \{x_0\}$, then

$$\begin{aligned} \int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \cdot \nabla \phi dx &= \int_{\Omega_1 \setminus B_\varepsilon} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \phi dx \\ &\quad + \int_{B_\varepsilon} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \phi dx \end{aligned} \quad (25)$$

where $B_\varepsilon \subset \Omega_1$ is a ball centered in x_0 with radius $\varepsilon > 0$.

Since $u_1 \in C^2(\Omega \setminus B_\varepsilon)$ and $|\nabla u_1| > 0$ in $\Omega \setminus B_\varepsilon$, it follows from the Divergence Theorem that

$$\begin{aligned} \int_{\partial\Omega_1 \cup \partial B_\varepsilon} \phi |\nabla u_1|^{p-2} \nabla u_1 \cdot \eta \, dS_x &= \\ &= \int_{\Omega_1 \setminus B_\varepsilon} \operatorname{div} \left(\phi |\nabla u_1|^{p-2} \nabla u_1 \right) dx \\ &= \int_{\Omega_1 \setminus B_\varepsilon} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \phi \, dx + \int_{\Omega_1 \setminus B_\varepsilon} \phi \operatorname{div} \left(|\nabla u_1|^{p-2} \nabla u_1 \right) dx \\ &= \int_{\Omega_1 \setminus B_\varepsilon} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \phi \, dx - \int_{\Omega_1 \setminus B_\varepsilon} \phi f_1(x, u_1, \nabla u_1) \, dx. \end{aligned}$$

Therefore,

$$\int_{\Omega_1 \setminus B_\varepsilon} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \phi \, dx = \int_{\Omega_1 \setminus B_\varepsilon} \phi f_1(x, u_1, \nabla u_1) \, dx + I_1 + I_2,$$

where

$$I_1 := \int_{\partial\Omega_1} \phi |\nabla u_1|^{p-2} \frac{\partial u_1}{\partial \eta} \, dS_x \leq 0$$

and

$$I_2 := - \int_{\partial B_\varepsilon} \phi |\nabla u_1|^{p-2} \frac{\partial u_1}{\partial \eta} \, dS_x.$$

The regularity of u_1 implies that $\left| |\nabla u_1|^{p-2} \frac{\partial u_1}{\partial \eta} \right| = |\nabla u_1|^{p-1} \leq C$ for some positive constant C which does not depend on u_1 . Thus,

$$|I_2| \leq C \|\phi\|_\infty |\partial B_\varepsilon| \rightarrow 0 \text{ (when } \varepsilon \rightarrow 0 \text{)}.$$

Consequently,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega_1 \setminus B_\varepsilon} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \phi \, dx &\leq \lim_{\varepsilon \rightarrow 0} \int_{\Omega_1 \setminus B_\varepsilon} \phi f_1(x, u_1, \nabla u_1) \, dx \\ &= \int_{\Omega_1} \phi f_1(x, u_1, \nabla u_1) \, dx. \end{aligned}$$

On the other hand,

$$\left| \int_{B_\varepsilon} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \phi \, dx \right| \leq \int_{B_\varepsilon} |\nabla u_1|^{p-1} |\nabla \phi| \, dx \leq C \|\nabla \phi\|_\infty |B_\varepsilon| \rightarrow 0,$$

when $\varepsilon \rightarrow 0$.

Now, by making $\varepsilon \rightarrow 0$ in (25) we obtain

$$\int_{\Omega} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \cdot \nabla \phi \, dx \leq \int_{\Omega_1} \phi f_1(x, u_1, \nabla u_1) \, dx \leq \int_{\Omega} \phi f(x, \underline{u}, \nabla \underline{u}) \, dx. \quad \square$$

Remark 13 (i) The hypothesis on Z_1 can be obtained if we suppose, for instance, $0 \leq f(x, t, v)$ and, for all $t > 0$, $\{(x, v) : f(x, t, v) = 0\}$ is a finite set. (Of course, the more interesting case occurs when $f(x, 0, v) = 0$.) (ii) See H. Lou [22] for further information on the singular set Z_1 .

Proof of the Theorem. From Remark 6 follows that

$$k_1(\Omega_2) \leq k_1(B_\rho).$$

So, if f satisfies the hypotheses (H1) and (H2), it also satisfies the hypotheses of Theorem 9. By applying Theorem 9, there exists a positive radial function $u_\rho \in C^{1,\tau}(\overline{B_\rho})$ such that

$$\begin{cases} -\Delta_p u_\rho &= \omega_\rho(|x - x_0|)f(u_\rho, |\nabla u_\rho|) & \text{in } B_\rho(x_0), \\ u_\rho &= 0 & \text{on } \partial B_\rho(x_0). \end{cases}$$

Moreover, the only critical point of u_ρ occurs at $x = x_0$.

It follows from Lemma 12 that

$$\underline{u}(x) = \begin{cases} u_\rho(x), & \text{if } x \in B_\rho, \\ 0, & \text{if } x \in \Omega \setminus B_\rho \end{cases}$$

is a sub-solution of problem (1).

Define

$$\overline{u} = M \frac{\phi_{\Omega_2}}{\|\phi_{\Omega_2}\|_\infty}.$$

Of course, $\overline{u} \leq M$ and $\|\nabla \overline{u}\|_\infty = M \frac{\|\nabla \phi_{\Omega_2}\|_\infty}{\|\phi_{\Omega_2}\|_\infty} \leq \gamma_\rho M$, by hypothesis. So, it follows from (H1) that

$$f(\overline{u}, |\nabla \overline{u}|) \leq k_1(\Omega_2)M^{p-1}.$$

Moreover,

$$-\Delta_p \overline{u} = -\Delta_p \left(M \frac{\phi_{\Omega_2}}{\|\phi_{\Omega_2}\|_\infty} \right) = k_1(\Omega_2)M^{p-1}\|\omega\|_\infty, \quad (26)$$

and then, by (H2)

$$-\Delta_p \overline{u} \geq f(\overline{u}, |\nabla \overline{u}|) \|\omega\|_\infty$$

and, since $\overline{u} > 0$ on $\partial\Omega$, \overline{u} is a super-solution of (1).

Moreover, the pair $(\underline{u}, \overline{u})$ is ordered. In fact, if $x \in \Omega \setminus B_\rho$ the result is immediate. Otherwise we know that,

$$\underline{u} = u_\rho \in C = \{u \in C^1(\overline{B_\rho}) : 0 \leq u \leq M, \text{ and } \|\nabla u\|_\infty \leq \gamma_\rho M\},$$

and therefore, by (H1), $f(u_\rho, |\nabla u_\rho|) \leq k_1(\Omega_2)M^{p-1}$ and then

$$-\Delta_p \underline{u} = \omega_\rho f(u_\rho, |\nabla u_\rho|) \leq k_1(\Omega_2)M^{p-1}\|\omega\|_\infty = -\Delta_p \left(M \frac{\phi_{\Omega_2}}{\|\phi_{\Omega_2}\|_\infty} \right) = -\Delta_p \overline{u}.$$

Moreover

$$u_\rho = 0 \leq M \frac{\phi_{\Omega_2}}{\|\phi_{\Omega_2}\|_\infty} = \overline{u}$$

on ∂B_ρ . We are done, since follows from the comparison principle that $\underline{u} \leq \overline{u}$ in $B_\rho \subset \Omega$. \square

6 Applications

In this section we choose two concrete domains Ω_2 for application of Theorem 10. In the first example, we consider a ball $B_R(x_1) = \Omega_2$ so that $\Omega \subset B_R$. In the second, we consider $\Omega_2 = \Omega$ and use a result by Payne and Philippin [24]. For this, we need to suppose that Ω is convex.

6.1 About the constants k_1 and k_2

Here we establish some results about the constants $k_1(\Omega)$ and $k_2(B_\rho)$ for $B_\rho \subset \Omega$. Our comments are based in remarks made in [5]. First we observe that, according to Remark 6, we have $k_1(\Omega_2) \leq k_1(\Omega_1)$, if $\Omega_1 \subset \Omega_2$.

Since

$$k_2(B_\rho) = \left[\max_{0 \leq r \leq \rho} \int_r^\rho \left(\int_0^r K(s, \theta) ds \right)^{\frac{1}{p-1}} d\theta \right]^{1-p},$$

is easy to conclude that $k_2(B_\rho) \rightarrow \infty$ if $\rho \rightarrow 0$. Also, larger values of ρ imply smaller values of $k_2(B_\rho)$ and, as we will see, also smaller values of γ_ρ . In this paper, for $x_0 \in \Omega$, we choose B_ρ as the largest ball centered at x_0 and contained in Ω .

By Lemma 7, we have $k_1(\Omega) \leq k_2(B_\rho)$ for all $B_\rho \subset \Omega$. Therefore

$$k_1(\Omega) \leq \Lambda := \inf\{k_2(B_\rho) : B_\rho \subset \Omega\}.$$

In the special case $\omega \equiv 1$ the constant Λ can be obtained since

$$k_2(B_\rho) = \left[\max_{0 \leq r \leq \rho} \left(\int_t^\rho \theta^{\frac{1-N}{p-1}} d\theta \right) \left(\int_0^t s^{N-1} ds \right)^{\frac{1}{p-1}} \right]^{1-p}.$$

In this situation we have

$$k_2(B_\rho) = \frac{C_{N,p}}{\rho^p}$$

where

$$C_{N,p} = \begin{cases} \frac{p^p}{(p-1)^{p-1}} e^{p-1} & \text{if } N = p; \\ \frac{N^p}{(p-1)^{p-1}} \left(\frac{p}{N}\right)^{\frac{p(p-1)}{p-N}} & \text{if } N \neq p. \end{cases} \quad (27)$$

In this case,

$$\Lambda = \frac{C_{n,p}}{\rho_*^p}, \quad \text{where } \rho_* := \sup\{\rho : B_\rho(x_0) \subset \Omega, x_0 \in \Omega\}.$$

6.2 Radial Supersolution

For all $x \in \Omega$, let $d(x) = \text{dist}(x, \partial\Omega)$. We denote by $r_* = \sup_{x \in \Omega} d(x)$. Let B_{r_*} be a ball with center at $x_0 \in \Omega$ such that $B_{r_*} \subset \Omega$.

Choose R such that $\Omega \subset B_R$, where B_R is a ball with center at $x_1 \in \Omega$ and let $\phi_R \in C^{1,\alpha}(\overline{B_R(x_1)}) \cap W_0^{1,p}(B_R(x_1))$ be the unique positive solution of

$$\begin{cases} -\Delta_p \phi_R &= \|\omega\|_\infty & \text{in } B_R(x_1), \\ \phi_R &= 0 & \text{on } \partial B_R(x_1), \end{cases} \quad (28)$$

and consider the positive constant $k_1(\Omega_2) = k_1(B_R) = \|\phi_R\|_\infty^{-(p-1)}$.

We define, as in Theorem 10,

$$\bar{u} := M \frac{\phi_R}{\|\phi_R\|_\infty} \in C^{1,\alpha}(\overline{B_R(x_1)}) \cap W_0^{1,p}(B_R(x_1)).$$

Of course, $0 < \bar{u} \leq M$. According to Section 4, we have

$$\begin{aligned} \phi_R(r) &= \int_r^R \varphi_q \left(\int_0^\theta \left(\frac{s}{\theta} \right)^{N-1} \|\omega\|_\infty ds \right) d\theta \\ &= \|\omega\|_\infty^{\frac{1}{p-1}} \int_r^R \varphi_q \left(\frac{1}{\theta^{N-1}} \int_0^\theta s^{N-1} ds \right) d\theta \\ &= \|\omega\|_\infty^{\frac{1}{p-1}} \int_r^R \left(\frac{\theta}{N} \right)^{\frac{1}{p-1}} d\theta \\ &= \frac{p-1}{p} \left(\frac{\|\omega\|_\infty}{N} \right)^{\frac{1}{p-1}} \left(R^{\frac{p}{p-1}} - r^{\frac{p}{p-1}} \right). \end{aligned} \quad (29)$$

On the other hand, we have $\nabla \phi_R(x) = \phi'_R(r) \frac{x-x_0}{r}$, from what follows $|\nabla \phi_R(x)| = |\phi'_R(r)|$. Thus,

$$\|\nabla \phi_R\|_\infty = |\phi'_R(R)| = \left(\int_0^R \left(\frac{s}{R} \right)^{N-1} \|\omega\|_\infty ds \right)^{\frac{1}{p-1}} = \left(\frac{\|\omega\|_\infty}{N} \right)^{\frac{1}{p-1}} R^{\frac{1}{p-1}}$$

and

$$\frac{\|\nabla \phi_R\|_\infty}{\|\phi_R\|_\infty} = \frac{p}{p-1} R^{\frac{1}{p-1} - \frac{p}{p-1}} = \frac{q}{R}. \quad (30)$$

So, we need to choose $\rho > 0$ such that $B_\rho \subset \Omega$ and

$$\frac{q}{R} < \gamma_\rho,$$

in order to have

$$0 \leq |\nabla \bar{u}| = M \frac{|\nabla \phi_R|}{\|\phi_R\|_\infty} \leq M \frac{\|\nabla \phi_R\|_\infty}{\|\phi_R\|_\infty} = \frac{q}{R} M \leq \gamma_\rho M.$$

To choose ρ , let us consider the possibilities

(i) $r_* \leq \frac{R}{q}$ ($< R$).

We choose $\rho = r_*$, because

$$\frac{\|\nabla\phi_R\|_\infty}{\|\phi_R\|_\infty} = \frac{q}{R} \leq \frac{1}{r_*} = \frac{1}{\rho} \leq \gamma_\rho.$$

(ii) $\frac{R}{q} \leq r_*$ ($< R$).

We choose $\rho = \frac{R}{q}$, since

$$\frac{\|\nabla\phi_R\|_\infty}{\|\phi_R\|_\infty} = \frac{q}{R} = \frac{1}{\rho} \leq \gamma_\rho.$$

In the special case $\omega_\rho \equiv 1$, we can always choose $\rho = r_*$, since

$$\frac{\|\nabla\phi_R\|_\infty}{\|\phi_R\|_\infty} = \frac{q}{R} \leq \frac{q}{r_*} = \frac{q}{\rho} = \gamma_\rho.$$

This value of ρ corresponds to the smallest values of $k_2(B_\rho)$ and γ_ρ . The best value for $k_1(B_R)$ is obtained when R is the smallest radius such that $B_R(x_1) \supset \Omega$ for $x_1 \in \Omega$.

6.3 Applying a maximum principle of Payne and Phillipin

If we choose $\Omega_2 = \Omega$, we need to suppose that Ω is convex to control the quotient (23). For this, we consider the torsional creep problem

$$\begin{cases} -\Delta_p \psi_\Omega &= 1 & \text{in } \Omega, \\ \psi_\Omega &= 0 & \text{on } \partial\Omega. \end{cases} \quad (31)$$

For more information on the torsional creep problem, see Kawohl [18].

In order to estimate the quotient (23), we state a maximum principle of Payne and Philippin [24]:

Theorem 14 *Let $\Omega \subset \mathbb{R}^N$ be a convex domain such that $\partial\Omega$ is a $C^{2,\alpha}$ surface. If $u = \text{const.}$ on $\partial\Omega$, then*

$$\Phi(x) = 2 \frac{p-1}{p} |\nabla \psi_\Omega|^p + 2\psi_\Omega \quad (32)$$

takes its maximum value at a critical point of ψ_Ω .

Lemma 15 *If Ω satisfies (H4), then*

$$\|\nabla \psi_\Omega\|_\infty \leq (q \|\psi_\Omega\|_\infty)^{\frac{1}{p}},$$

what yields

$$\frac{\|\nabla \psi_\Omega\|_\infty}{\|\psi_\Omega\|_\infty} \leq \frac{q^{\frac{1}{p}}}{\|\psi_\Omega\|_\infty^{\frac{1}{q}}}.$$

Proof. By Theorem 14, Φ takes its maximum value at a point where $\nabla\psi_\Omega = 0$. So, it follows from (32) that

$$2\frac{p-1}{p}|\nabla\psi_\Omega|^p + 2\psi_\Omega \leq 2\|\psi_\Omega\|_\infty.$$

Therefore

$$|\nabla\psi_\Omega|^p \leq \frac{p}{p-1} (\|\psi_\Omega\|_\infty - \psi_\Omega(x)) \leq \frac{p}{p-1} \|\psi_\Omega\|_\infty = q\|\psi_\Omega\|_\infty, \quad \forall x \in \Omega,$$

thus producing

$$|\nabla\psi_\Omega| \leq (q\|\psi_\Omega\|_\infty)^{\frac{1}{p}}, \quad \forall x \in \Omega.$$

But the p -Laplacian is degenerated at the origin. So, in order to estimate the quotient (23), a regularization of $-\Delta_p$ is done by considering, as in Sakaguchi [26], the problem

$$\begin{cases} -\operatorname{div}((\varepsilon + |\nabla\phi_\varepsilon|^2)^{\frac{p-2}{2}} \nabla\phi_\varepsilon) &= 1 & \text{in } \Omega, \\ \phi_\varepsilon &= 0 & \text{on } \partial\Omega. \end{cases}$$

Sakaguchi proves that the solution ϕ_ε converges uniformly to ψ_Ω as $\varepsilon \rightarrow 0$. The regularization permits us to estimate (23) in the case of the torsional creep problem (31):

$$\frac{\|\nabla\psi_\Omega\|_\infty}{\|\psi_\Omega\|_\infty} \leq \frac{q^{\frac{1}{p}}}{\|\psi_\Omega\|_\infty^{1-\frac{1}{p}}} = \frac{q^{\frac{1}{p}}}{\|\psi_\Omega\|_\infty^{\frac{1}{q}}}. \quad \square$$

An immediate consequence of Lemma 15 is an estimate of the quotient (23) in the case $\Omega = \Omega_2$: we have

$$\frac{\|\nabla\phi_\Omega\|_\infty}{\|\phi_\Omega\|_\infty} \leq \frac{(q\|\omega\|_\infty)^{\frac{1}{p}}}{\|\phi_\Omega\|_\infty^{\frac{1}{q}}}. \quad (33)$$

In fact, if ψ_Ω is a solution of the torsional creep problem (31), then

$$\phi_\Omega = \|\omega\|_\infty^{\frac{1}{p-1}} \psi_\Omega$$

is a solution of (21). So,

$$\frac{\|\nabla\phi_\Omega\|_\infty}{\|\phi_\Omega\|_\infty} = \frac{\|\nabla\psi_\Omega\|_\infty}{\|\psi_\Omega\|_\infty} \leq \frac{q^{\frac{1}{p}}}{\|\psi_\Omega\|_\infty^{\frac{1}{q}}} = \frac{q^{\frac{1}{p}}}{\|\phi_\Omega\|_\infty^{\frac{1}{q}}} \|\omega\|_\infty^{\frac{1}{p}} = \frac{(q\|\omega\|_\infty)^{\frac{1}{p}}}{\|\phi_\Omega\|_\infty^{\frac{1}{q}}}. \quad (34)$$

We observe that the quotient (23) was controlled for any convex domain $\Omega_2 \supset \Omega$.

As in the Subsection 6.2, let B_{r_*} be a ball with larger radius such that $B_{r_*} \subset \Omega$. We consider the solution ϕ_* of the problem

$$\begin{cases} -\Delta_p \phi_* &= \|\omega\|_\infty & \text{in } B_{r_*}, \\ \phi_* &= 0 & \text{on } \partial B_{r_*}. \end{cases}$$

Since $B_{r_*} \subset \Omega$, from the comparison principle follows that

$$\|\phi_*\|_\infty \leq \|\phi_\Omega\|_\infty.$$

But

$$\begin{aligned} \|\phi_*\|_\infty &= \int_0^{r_*} \left(\int_0^\theta \left(\frac{s}{\theta} \right)^{N-1} \|\omega\|_\infty ds \right)^{\frac{1}{p-1}} d\theta \\ &= \left(\frac{\|\omega\|_\infty}{N} \right)^{\frac{1}{p-1}} \int_0^{r_*} \theta^{\frac{1}{p-1}} d\theta = \left(\frac{\|\omega\|_\infty}{N} \right)^{\frac{q}{p}} \frac{r_*^q}{q}, \end{aligned}$$

thus yielding

$$\frac{\|\nabla \phi_\Omega\|_\infty}{\|\phi_\Omega\|_\infty} \leq \frac{(q\|\omega\|_\infty)^{\frac{1}{p}}}{\|\phi_\Omega\|_\infty^{\frac{1}{q}}} \leq \frac{q^{\frac{1}{p} + \frac{1}{q}}}{r_*} \|\omega\|_\infty^{\frac{1}{p}} \left(\frac{N}{\|\omega\|_\infty} \right)^{\frac{1}{p}} = \sqrt[p]{N} \frac{q}{r_*}.$$

We now choose ρ given by

$$\rho = \frac{r_*}{q \sqrt[p]{N}} = \frac{p-1}{p \sqrt[p]{N}} r_* \quad (< r_*).$$

Then, we have

$$\frac{\|\nabla \phi_\Omega\|_\infty}{\|\phi_\Omega\|_\infty} \leq \frac{1}{\rho} = \gamma_\rho.$$

In the special case $\omega \equiv 1$, we can take ρ such that

$$\frac{q}{\rho} = \frac{q \sqrt[p]{N}}{r_*},$$

since $\gamma_\rho = q/\rho$.

Thus, we have

$$\rho = \frac{r_*}{\sqrt[p]{N}} < r_*$$

and

$$\frac{\|\nabla \phi_\Omega\|_\infty}{\|\phi_\Omega\|_\infty} \leq \frac{q}{\rho} = \gamma_\rho.$$

7 Examples

In this section, we start by studying the problem

$$\begin{cases} -\Delta_p u &= \lambda u(x)^{q-1} (1 + |\nabla u(x)|^p) & \text{em } \Omega, \\ u &= 0 & \text{em } \partial\Omega, \end{cases} \quad (35)$$

where Ω is a smooth, bounded domain in \mathbb{R}^N , $1 < q < p$, and λ a positive parameter. Problem (35) is sublinear at the origin.

The solution of this example will permit us to solve

$$\begin{cases} -\Delta_p u &= \lambda f(u, |\nabla u(x)|) & \text{em } \Omega, \\ u &= 0 & \text{em } \partial\Omega, \end{cases}$$

for a class of nonlinearities f that are superlinear both at the origin and at $+\infty$.

Results of Huang ([15]) guarantee the unicity of solutions in

$$\Gamma_q = \left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} |u|^q = 1 \right\}$$

for the problem

$$\begin{cases} -\Delta_p u &= \lambda u(x)^{q-1} & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{cases}$$

In Montenegro and Montenegro ([23]) degree theory and the method of sub- and supersolutions are combined to present conditions for existence and nonexistence of weak, positive solutions for the problem

$$\begin{cases} -\Delta_p u &= \frac{a}{1+ku} |\nabla u|^p + b(1+ku)^{p-1} & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

where a e b are positive constants and $k \geq 0$.

Also, in Iturriaga, Lorca and Sanchez ([17]) no qual os autores consideram o problema

$$\begin{cases} -\Delta_p u &= \lambda f(x, u) + |\nabla u|^p & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \quad (36)$$

where λ is a positive parameter and $f(x, u)$ a Caratheodory function such that

$$c_0 u^{q-1} \leq f(x, u) \leq c_1 u^{q-1}, \text{ para todo } (x, t) \in \overline{\Omega} \times [0, \infty)$$

for positive constants c_0, c_1 . However, the problem was solved by applying a change of variables that transforms (36) into a problem that does not depend on the gradient.

Under similar but different hypotheses, problem (35) was solved in [6]. Here, we show that this problem has a positive solution for each $\lambda \in (0, \lambda^*]$, where λ^* will be given in the sequence. To solve problem (35) we consider, as in Subsection 6.2, B_ρ as the largest open ball contained in Ω and B_R such that $\Omega \subseteq B_R$.

Remark 16 The nonlinearity $\lambda u(x)^{q-1}(1 + |\nabla u(x)|^p)$ is related to the weight function $\omega(x) \equiv 1$. So, the constants in hypotheses (H1) e (H2) are given by

$$k_1 := k_1(B_R) = \|\phi_R\|_{\infty}^{-(p-1)} = \left(\frac{p-1}{p} \right)^{1-p} N R^{-p}, \quad (37)$$

$$k_2 := k_2(B_\rho) = \begin{cases} \left[\frac{p-1}{p} \left(\frac{p}{N} \right)^{\frac{N}{N-p}} \right]^{1-p} \frac{N}{\rho^p}, & \text{if } N \neq p, \\ \left(\frac{p-1}{ep} \right)^{1-p} \frac{p}{\rho^p}, & \text{if } N = p, \end{cases} \quad (38)$$

and

$$\gamma = \gamma_\rho = \frac{p}{p-1} \frac{1}{\rho}. \quad (39)$$

From now on, k_1 and k_2 denote the constants (37) and (38), respectively. According to Lemma 7, we have $k_1 < k_2$. \triangleleft

Of course, the nonlinearity $\lambda u(x)^{q-1}(1+|\nabla u(x)|^p)$ satisfies (H3) for any value of λ . We will obtain constants δ, M (with $0 < \delta < M$) such that hypotheses (H1) and (H2) of Theorem 10 are verified.

To satisfy (H1), M must be chosen such that

$$\lambda M^{q-1}(1 + (\mu M)^p) \leq \alpha M^{p-1}. \quad (40)$$

So, defining the function $H: [0, \infty) \rightarrow [0, \infty]$ by $H(M) = M^{q-p}(1 + \mu^p M^p)$, the last inequality is equivalent to

$$H(M) \leq \frac{\alpha}{\lambda}.$$

We have

$$\lim_{M \rightarrow 0^+} H(M) = \infty = \lim_{M \rightarrow \infty} H(M),$$

and the function H has a unique critical point M_* , given by

$$\mu^p M_*^p = \frac{p}{q} - 1,$$

where H assumes its minimum value

$$H(M_*) = M_*^{q-p}(1 + \mu^p M_*^p) = \frac{1}{\mu^{q-p}} \left(\frac{p}{q} - 1 \right)^{\frac{q-p}{p}} \left(\frac{p}{q} \right).$$

Taking $M := M_*$ in (40) and defining

$$\lambda^* = \frac{k_1}{H(M_*)}, \quad (41)$$

hypothesis (H1) is verified for any $0 < \lambda \leq \lambda^*$. The choice $M = M_*$ makes λ^* to be the best possible value of the parameter such that Theorem 10 guarantees the existence of a positive solution for problem (35).

Now, for any fixed $\lambda \in (0, \lambda^*]$, we try to verify (H2). More precisely, we look for $\delta_\lambda := \delta(\lambda)$ such that

$$\lambda u^{q-1}(1 + |\nabla u|^p) \geq k_2 \delta_\lambda^{p-1}, \quad \delta_\lambda \leq u \leq M_*, \quad 0 \leq |\nabla u| \leq \gamma M_*.$$

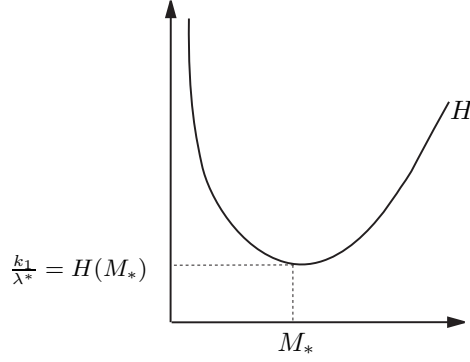


Figure 2: The graph of H assumes its minimum at the point M^* .

For this, we consider the function $G: (0, \infty) \rightarrow [0, \infty)$ given by

$$G(x) = x^{q-p}. \quad (42)$$

We clearly have $G(x) \leq H(x)$ for any $x \in (0, \infty)$ and (H2) is verified if

$$\lambda G(\delta_\lambda) \geq k_2, \quad (43)$$

that is,

$$\delta_\lambda \leq \left(\frac{\lambda}{k_2} \right)^{\frac{1}{p-q}}. \quad (44)$$

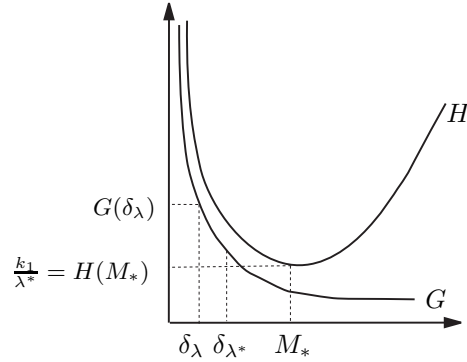


Figure 3: The graphs of H and G .

So, for any $\lambda \in (0, \lambda^*]$, (H2) is satisfied if we take $\delta_\lambda > 0$ verifying the above inequality. Observe that the same value of δ_λ is valid for any $\tilde{\lambda} \in [\lambda, \lambda^*]$.

Since $0 < \lambda \leq \lambda^*$, the largest value of δ_λ is attained at λ^* . So, the condition

$\delta_\lambda < M_*$ always holds:

$$\delta_\lambda \leq \left(\frac{\lambda^*}{k_2}\right)^{\frac{1}{p-q}} \leq \left(\frac{\lambda^*}{k_1}\right)^{\frac{1}{p-q}} = \left(\frac{1}{H(M_*)}\right)^{\frac{1}{p-q}} = M_* \left(\frac{q}{p}\right)^{\frac{1}{p-q}} < M_*.$$

Remark 17 The existence of positive solutions for the problem

$$\begin{cases} -\Delta_p u &= \lambda \omega(x) u(x)^{q-1} (1 + |\nabla u(x)|^p) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \quad (45)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded, smooth domain and $1 < q < p$ follows analogously for any continuous weight function $\omega: \bar{\Omega} \rightarrow \mathbb{R}$. We can also change p for any value $0 < \theta < p$.

The main advantage of the method that leads to Theorem 10 are the hypotheses (H1) and (H2).

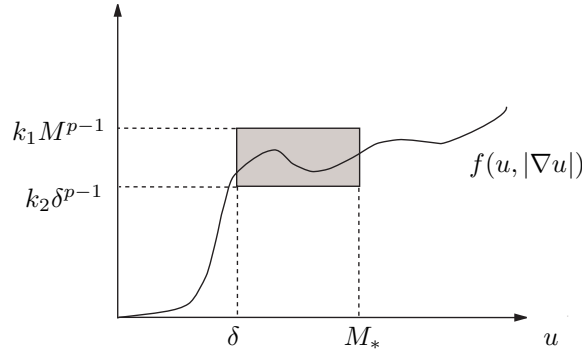


Figure 4: Example of a nonlinearity f , superlinear at the origin and satisfying (H1) e (H2). O graph illustrates the perspective $|\nabla u| = \text{constant}$.

With the exception of (H3), no other assumption on the nonlinearity f is necessary. So, f can be superlinear both at the origin and at $+\infty$.

Remark 18 The radial problem

$$\begin{cases} -\Delta_p u &= \lambda \omega(|x|) u(x)^r (1 + |\nabla u(x)|^\theta), & \text{if } R_1 < |x| < R_2, \\ u &= 0, & \text{if } |x| = R_1 \text{ or } |x| = R_2, \end{cases} \quad (46)$$

where $\lambda = 1$, $r > p - 1$, $0 \leq \theta \leq p$ and $\omega: [R_1, R_2] \rightarrow [0, \infty)$ is a continuous function not equal zero was considered in [11]. In that work, the existence of a positive solution for (46) was obtained as a consequence of Krasnosel'skii Fixed Point Theorem for mappings defined in cones.

Our result complements those obtained in [11] and guarantees the existence of a solution for (46) in the case $0 < r < p - 1$ and $0 < \theta \leq p$.

In fact, if $\lambda \leq \lambda^*$, the existence of a positive solution follows from Theorem 10. Since $\omega \neq 0$, there exists B_ρ contained in the domain where $\omega > 0$. In B_ρ we obtain a subsolution \underline{u} of (46).

The inclusion of the parameter λ is necessary because of the hypotheses (H1) and (H2). In the particular case $\lambda = 1$, the existence of solution is obtained only if $1 \leq \lambda^*$.

8 Appendix

In this appendix we give a direct prove that the radial operator T of Section 4 is continuous and compact.

In fact, compactness of T can be obtained from Arzelá-Ascoli and Dominated Convergence Theorems.

Let $\{u_m\}_{m \in \mathbb{N}}$ a bounded sequence in X , $\|u_m\| \leq M$. In this case, $\{Tu_m\}$ and $\{(Tu_m)'\}$ are bounded in X and uniformly equicontinuous.

In fact, let $C = \sup_{0 \leq t, s \leq M} f(s, t)$. Since $K(s, \theta) = \left(\frac{s}{\theta}\right)^{N-1} \omega_\rho(s) \leq \omega_\rho(s)$ and $0 \leq \theta \leq \rho$, we have

$$Tu_m(r) \leq \int_0^\rho \varphi_q \left(C \int_0^\theta K(s, \theta) ds \right) d\theta \leq \rho \varphi_q \left(C \int_0^\rho \omega_\rho(s) ds \right),$$

showing that $Tu_m(r)$ is uniformly bounded in the sup norm.

We also have,

$$\begin{aligned} |(Tu_m)'(r)| &= \varphi_q \left(\int_0^r K(s, r) f(u_m(s), |\nabla u_m(s)|) ds \right) \\ &\leq \varphi_q \left(C \int_0^\rho \omega_\rho(s) ds \right), \end{aligned}$$

proving that $\{Tu_m\}$ is equicontinuous.

Let us prove that $\{(Tu_m)'\}$ is equicontinuous. Deriving $(Tu_m)'(r)$ we have

$$|(Tu_m)''(r)| = \left(\frac{1}{p-1} \right) \left[|v_m'(r)|^{2-p} w_\rho(s) f(u_m(s), |u_m'(s)|) + \left(\frac{n-1}{r} \right) |v_m'(r)| \right],$$

where $v_m = (Tu_m)'$.

If $1 < p \leq 2$, the right-hand side of the last equality is bounded, thus $(Tu_m)'$ is Lipschitz-continuous and, consequently, equicontinuous.

If $p > 2$, $|(Tu_m)'(r)|$ is Hölder continuous with exponent $\frac{1}{p-1}$ (and consequently $|(Tu_m)(r)|$ is equicontinuous). In fact, $(Tu_m)'(r) = (\varphi_q \circ \lambda_m)(r)$, where

$$\varphi_q(x) = x^{\frac{1}{p-1}}$$

is a Hölder continuous function and

$$\lambda_m(r) := \int_0^r K(s, r) f(u_m(s), |\nabla u_m(s)|) ds.$$

We claim that λ_m is locally Lipschitz continuous, uniformly on m . For this, we note that $\lambda_m \in C^1([0, R])$ with

$$\lim_{r \rightarrow 0^+} \lambda_m(r) = 0$$

and

$$\lim_{r \rightarrow 0^+} \lambda'_m(r) = \frac{\omega(0) f(u_m(0), |u'_m(0)|)}{N}.$$

Therefore, the Mean Value Theorem guarantees the existence of $L > 0$ such that

$$|\lambda_m(r) - \lambda_m(t)| \leq L|r - t|,$$

proving our claim.

Since $(Tu_m)'(r) = \varphi_q \circ \lambda_m(r)$ we can conclude the equicontinuity of $\{(Tu_m)'\}$

$$\begin{aligned} |(Tu_m)'(r) - (Tu_m)'(t)| &\leq |\varphi_q(\lambda_m(r)) - \varphi_q(\lambda_m(t))| \\ &\leq |\lambda_m(r) - \lambda_m(t)|^{\frac{1}{p-1}} \leq L|r - t|^{\frac{1}{p-1}}. \end{aligned}$$

We also note that, if $\{u_m\}_{m \in \mathbb{N}}$ converges uniformly to u in $[0, R]$, then $Tu_{m_j} \rightarrow Tu$ for all the subsequences $\{u_{m_j}\}$ of $\{u_m\}$, by the Dominated Convergence Theorem. From this follows that T is continuous. \square

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